CHAPTER 10

FURTHER TECHNIQUES AND APPLICATIONS OF INTEGRATION

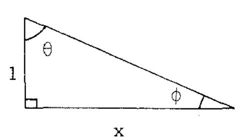
10.1 Trigonometric Integrals

PREREQUISITES

1. Recall how to differentiate and integrate trigonometric functions (Sections 5.2 and 7.1).
2. Recall how to integrate by substitution (Section 7.2).
3. Recall how to define trigonometric functions in terms of the sides of a right triangle (Section 5.1).
4. Recall how to complete a square (Section R.1).

PREREQUISITE QUIZ

1. Differentiate $y = \sin 4x - \cos^2 x + \tan x$.
2. (a) Evaluate $\int \cos 3x\, dx$.

   (b) Evaluate $\int \sin(2x - 3)dx$.
3. 

   Consider the figure at the left.

   (a) What is $\sin \phi$?

   (b) What is $\cos \theta$?
4. Complete the square in the following:

   (a) $x^2 + 2x + 4$

   (b) $x^2 - 3x - 1$

GOALS

1. Be able to use trigonometric identities for integrating expressions involving products of sines and cosines.

2. Be able to use trigonometric substitution for integrating expressions involving $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$, $\sqrt{a^2 + x^2}$, or $a^2 + x^2$.

STUDY HINTS

1. Half-angle formulas. You should memorize or learn to derive $\sin^2 x = (1 - \cos 2x)/2$ and $\cos^2 x = (1 + \cos 2x)/2$. If you forget which sign goes with which formula, substitute $x = 0$ as a check. These formulas can be derived by using $\cos 2x = \cos^2 x - \sin^2 x$ and $1 = \cos^2 x + \sin^2 x$. Adding and subtracting yields the desired results. The half-angle formulas are commonly used for integration.

2. Integrating $\sin^m x \cos^n x$. Basically, if one exponent is odd, use the identity $\cos^2 x + \sin^2 x = 1$ and substitute $u = \sin x$ or $u = \cos x$, whichever is the odd power . If both are even, use the half-angle formulas. The trigonometric integal box on p. 458 is a good one to know.

3. Substituting $u = \sec x$. When $\tan x$ and $\sec x$ appear, it is sometimes useful to substitute $u = \sec x$. Often, it is necessary to rewrite the integrand into a useful form. See Example 3(c).

4. Addition and product formulas. Knowing that $\sin 2x = 2 \sin x \cos x$ and that $\cos 2x = \cos^2 x - \sin^2 x$ may help you recall formulas (1a) and (1b) on p. 460. Simply substitute x for y . The addition formulas, along with $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$, are useful for deriving the product formulas. If you choose to memorize the product formulas, which are useful for integration, note that the angle $x-y$ always appears first as it is written on p. 460. Also, it may be

4. (continued)

helpful to ask yourself, "What if $x = y$?" In addition, note that only $\sin x \cos y$ has sine terms on the right-hand side.

5. Trigonometric substitutions. This technique is often used when $(\pm a^2 \pm x^2)^{n/2}$ appears in the integrand, where n = integer and a are constants. Notice that this technique is based upon the Pythagorean identities $\cos^2 x + \sin^2 x = 1$ and $\sec^2 x = 1 + \tan^2 x$. Know what substitution to use for x in each of the three cases. Using the substitution equation, draw an appropriate triangle and label it like those in the box on p. 461. After the integration is completed, use the triangle to express your answer in the original variable.

6. Integrating $\sec x$ and $\csc x$. Just note the interesting trick used in the integration on p. 462 (lines 3 - 5).

7. Integrating $\sec^3 x$. Often, when $\sqrt{1 + x^2}$ appears in the integrand, trigonometric substitution will call for the integration of $\sec^3 x$. The technique is shown in the solution to Example 8(a), p. 496. Note that $\sec^3 x$ is integrated by parts.

8. Integrals involving $ax^2 + bx + c$. In many instances, the first step is to complete the square. Then, use a trigonometric substitution.

9. Example 5 comment. If $a = \pm b$, remember that $\cos(0) = 1$.

10. Practice. A lot of material has been covered in this section. Items placed in memory are easy to forget. Practice helps to reinforce what has been memorized.

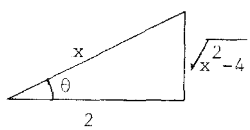
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Using the identity, $\sin^2 x = 1 - \cos^2 x$, we get $\int \sin^3 x \cos^3 x\,dx = \int (1 - \cos^2 x)\sin x \cos^3 x\,dx$. Now, substitute $u = \cos x$, so $du = -\sin x$, and the integral becomes $-\int (1 - u^2)u^3 du = \int (u^5 - u^3) du = u^6/6 - u^4/4 + C = \cos^6 x/6 - \cos^4 x/4 + C$.

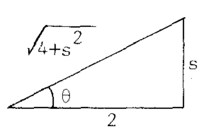
5. The half-angle formula $\cos^2 x = (1 + \cos 2x)/2$ gives us $\cos 2x - \cos^2 x = (\cos 2x - 1)/2$. Thus, $\int (\cos 2x - \cos^2 x) dx = \int (\cos 2x - 1) dx/2 = \sin 2x/4 - x/2 + C$.

9. Using the product formula $\sin x \sin y = (1/2)[\cos(x - y) - \cos(x + y)]$, we get $\int \sin 4x \sin 2x\,dx = \int (\cos 2x - \cos 6x)/2 = \sin 2x/4 - \sin 6x/12 + C$.

13. By using the hint, $\int \tan^3 x \sec^3 x\,dx$ becomes $\int (\sin^3 x/\cos^6 x) dx = \int [\sin x(1 - \cos^2 x)/\cos^6 x] dx$. Now, let $u = \cos x$, so $du = -\sin x\,dx$; therefore, the integral becomes $-\int [(1 - u^2)/u^6] du = \int (u^{-4} - u^{-6}) du = u^{-3}/(-3) - u^{-5}/(-5) + C = -1/3 \cos^3 x + 1/5 \cos^5 x + C = -\sec^3 x/3 + \sec^5 x/5 + C$.

17. 

Let $x = 2 \sec\theta$, so $dx = 2 \sec\theta \tan\theta\,d\theta$. Then $\int (\sqrt{x^2 - 4}/x) dx = \int (2\sqrt{\sec^2\theta - 1}/2 \sec\theta) \times 2 \sec\theta \tan\theta\,d\theta = 2\int \tan^2\theta\,d\theta = 2\int (\sin^2\theta/\cos^2\theta) d\theta = 2\int [(1 - \cos^2\theta)/\cos^2\theta] d\theta = 2\int (\sec^2\theta - 1) d\theta = 2(\tan\theta - \theta) + C$. We used the identities $\tan\theta = \sin\theta/\cos\theta$ and $\sin^2\theta + \cos^2\theta = 1$. Now, use the figure to get $2(\tan\theta - \theta) + C = 2(\sqrt{x^2 - 4}/2 - \cos^{-1}(2/x)) + C$.

21. 

Let $s = 2 \tan\theta$, so $ds = 2 \sec^2\theta\,d\theta$. Then $\int (s/\sqrt{4 + s^2}) ds$ becomes $\int [2 \tan\theta/\sqrt{4(1 + \tan^2\theta)}] \times 2 \sec^2\theta\,d\theta = 2\int \tan\theta \sec\theta\,d\theta = 2 \sec\theta + C$. From the figure, we get the final answer, $\sqrt{4 + s^2} + C$.

25. Completing the square, we have $4x^2 + x + 1 = 4(x + 1/8)^2 + (-1/16 + 1) = 4(x + 1/8)^2 + 15/16$. Let $u = x + 1/8$, so $du = dx$. Then, $\int(dx/\sqrt{4x^2 + x + 1}) = \int(du/\sqrt{4u^2 + 15/16})$. Factoring out $1/\sqrt{15/16}$ and letting $\theta = 8u/\sqrt{15}$, we have $\int(dx/\sqrt{4x^2 + x + 1}) = \int(2\ d\theta/\sqrt{\theta^2 + 1}) = 2 \sinh^{-1}\theta + C = (1/2) \sinh^{-1}(8u/\sqrt{15}) + C = (1/2) \times \sinh^{-1}((8x + 1)/\sqrt{15}) + C$.

29. Note that $\cos(\pi + x) = -\cos x$, so if n is odd, $\cos^n(\pi + x) = -\cos^n x$. Hence, $\int_0^{2\pi} \cos^n x\ dx = \int_0^{\pi} \cos^n x\ dx + \int_{\pi}^{2\pi} \cos^n x\ dx = \int_0^{\pi} \cos^n x\ dx + \int_0^{\pi} \cos^n(x + \pi)dx = \int_0^{\pi}(\cos^n x - \cos^n x)dx = \int_0^{2\pi} 0\ dx = 0$. Therefore, we need to consider only even values of n for four cases.

(a) For $n = 0$, $(1/2\pi)\int_0^{2\pi} \cos^0 x\ dx = (1/2\pi)\int_0^{2\pi} 1\cdot dx = 2\pi/2\pi = 1$.

(b) For $n = 2$, $(1/2\pi)\int_0^{2\pi} \cos^2 x\ dx = (1/4\pi)\int_0^{2\pi}(1 + \cos 2x)dx = (1/4\pi)(x + (1/2)\sin 2x)\big|_0^{2\pi} = 1/2$.

(c) For $n = 4$, $(1/2\pi)\int_0^{2\pi} \cos^4 x\ dx = (1/2\pi)\int_0^{2\pi}(\cos^2 x)^2 dx = (1/8\pi)\int_0^{2\pi}(1 + \cos 2x)^2 dx = (1/8\pi)\int_0^{2\pi}(1 + 2 \cos 2x + \cos^2 2x)dx = (1/8\pi)\int_0^{2\pi}[1 + 2 \cos 2x + (1/2)(1 + \cos 4x)]dx = (1/8\pi)(x + \sin 2x + x/2 + (1/8)\sin 4x)\big|_0^{2\pi} = 3/8$.

(d) For $n = 6$, $(1/2\pi)\int_0^{2\pi}\cos^6 x\ dx = (1/2\pi)\int_0^{2\pi}(\cos^2 x)^3 dx = \int_0^{2\pi}(1/(16\pi))(1 + \cos 2x)^3 dx = (1/(16\pi))\int_0^{2\pi}(1 + 3 \cos 2x + 3 \cos^2 2x + \cos^3 2x)dx = (1/16\pi)\int_0^{2\pi}(1 + 3 \cos 2x + (3/2)(1 + \cos 4x) + (1 - \sin^2 2x)\cos 2x)dx = (1/16\pi)(x + (3/2)\sin 2x + (3/2)x + (3/8)\sin 4x)\big|_0^{2\pi} + (1/16\pi)\int_0^{2\pi}(\cos 2x - \cos 2x \sin^2 2x)dx = (1/16\pi)[(5\pi) + ((1/2)\sin 2x - (1/6)\sin^3 2x)\big|_0^{2\pi}] = 5/16$.

33. Substitute $i = 10 \sin(377t)$ and a half-angle formula to get $P = (1/T)\int_0^T R\cdot 100 \sin^2(377t)dt = (50R/T)\int_0^T(1 - \cos(754t))dt = (50R/T)(t - (1/754)\sin(754t))\big|_0^T = (50R/T)(T - (1/754)\sin(754T)) = 50R - (50R/754T) \times \sin(754T)$. Now, substitute $R = 2.5$ and $T = 2\pi/377$ to get $P = 125 - (125\cdot 377/(2\pi\cdot 754)) \sin 4\pi = 125$.

37. (a) Differentiation yields $(d/dt)[(S(t))^3] = 3(S(t))^2 S'(t) = 3(t \sin t + \sin^2 t \cos^2 t)$. Now, integrate to get $(S(t))^3 = 3\int_c^t (x \sin x + \sin^2 x \cos^2 x)dx$. Substitute $t = 0$ into the given equation to get $(S(0))^2 S'(0) = 0 = S(0)$. Thus, $S(0)^3 = 0 = 3\int_c^0 (x \sin x + \sin^2 x \cos^2 x)dx$, so $c = 0$. Therefore, $(S(t))^3 = 3\int_0^t (x \sin x + \sin^2 x \cos^2 x)dx$.

(b) Let $I(t) = \int_0^t (x \sin x + \sin^2 x \cos^2 x)dx$. Let $A(t) = \int_0^t x \sin x \, dx$ and let $B(t) = \int_0^t \sin^2 x \cos^2 x \, dx$. Then $I(t) = A(t) + B(t)$. In $A(t)$, let $u = x$ so $du = dx$, and let $dv = \sin x \, dx$, so $v = -\cos x$. Then $A(t) = -x \cos x\Big|_0^t + \int_0^t \cos x \, dx = (-x \cos x + \sin x)\Big|_0^t = -t \cos t + \sin t$. Since $\sin x \cos x = (1/2) \sin 2x$, $B(t) = (1/4)\int_0^t \sin^2 2x \, dx = (1/8) \times \int_0^t (1 - \cos 4x)dx = (1/8)(x - (1/4)\sin 4x)\Big|_0^t = (1/8)/(t - (1/4)\sin 4t)$ Then $I(t) = -t \cos t + \sin t + t/8 - (1/32) \sin 4t$. Thus, $S(t) = \sqrt[3]{3I(t)} = [3(-t \cos t + \sin t + t/8 - (1/32)\sin 4t)]^{1/3}$.

(c) $S'(t) = 0$ whenever $t \sin t + \sin^2 t \cos^2 t = 0$; i.e., whenever $\sin t = 0$ or $t + \sin t \cos^2 t = 0$. Since $-1 \leqslant \sin t \leqslant 1$ and $-1 \leqslant \cos \leqslant 1$, $-1 < \sin t \cos^2 t < 1$. Thus, if $t > 1$, $t + \sin^2 t \cos^2 t > 1 - 1 = 0$. Therefore, the only zeros of $S'(t)$ for $t > 1$ are when $\sin t = 0$, i.e., $t = n\pi$. These are critical values of t, corresponding to relative maxima and minima of $S(t)$. When n is odd, $\sin t$, and correspondingly, $S'(t)$ changes from positive to negative, indicating a relative maximum excursion. There is no absolute maximum, since $S(t + 2\pi) > S(t)$.

SECTION QUIZ

1. Use the technique of trigonometric substitution to evaluate $\int \left(dx/\sqrt{1 - x^2} \right)$. Did you get the expected result?

2. Evaluate the following integrals:

   (a) $\int \cos^7\theta \, d\theta$

   (b) $\int \cos^6 t \, dt$

   (c) $\int \sec^{97} x \tan x \, dx$

   (d) $\int \left[\sin(3x/2)\cos 2x + 1/\sqrt{x^2 - 1} \right] dx$

   (e) $\int \sin 3x \sin 4x \, dx$

   (f) $\int [3/(1 + t^2)^{3/2}] \, dt$

3. Find the center of mass of the region bounded by $y = x \cos^2 x$, the x-axis, and the lines $x = 0$, $x = \pi/2$. [Hint: You may find it helpful to integrate $x^2 \cos \alpha x$ for a constant α.]

4. Evaluate the following integrals:

   (a) $\int \left[(3x + 5)/\sqrt{x^2 + 4x - 1} \right] dx$

   (b) $\int [(4x + 2)/(x^2 + 6x + 10)^{1/2}] \, dx$

   (c) $\int [(x - 1)/(x^2 - 2x - 5)] \, dx$

5. Underwater divers have recently discovered a sea monster at the ocean floor. Its shape has been described as follows: The region between $(4 - x^2)^{1/4}$ and $\sin x$ on $[0, \pi/4]$, revolved around the x-axis. If each unit represents 10 meters and the sea monster has a density of 1500 kg/m^3, how much does it weigh?

ANSWERS TO PREREQUISITE QUIZ

1. $4 \cos 4x + 2 \sin x \cos x + \sec^2 x$

2. (a) $\sin 3x/3 + C$

   (b) $-\cos(2x - 3)/2 + C$

3. (a) $1/\sqrt{1+x^2}$

   (b) $1/\sqrt{1+x^2}$

4. (a) $(x+1)^2+3$

   (b) $(x-3/2)^2-13/4$

ANSWERS TO SECTION QUIZ

1. $\sin^{-1}x + C$, as expected

2. (a) $\sin\theta - \sin^3\theta + 3\sin^5\theta/5 - \sin^7\theta/7 + C$

   (b) $5t/16 + \sin 2t/4 + 3\sin 4t/64 - \sin^3 2t/48 + C$

   (c) $\sec^{97}x/97 + C$

   (d) $\cos(x/2) - \cos(7x/2)/7 + \ln\left|x + \sqrt{x^2-1}\right| + C$

   (e) $\sin x/2 - \sin 7x/14 + C$

   (f) $3t/\sqrt{1+t^2} + C$

3. $(\pi(\pi^2-6)/3(\pi^2-4), \pi(2\pi^2-15)/16(\pi^2-4))$

4. (a) $3\sqrt{x^2+4x-1} - \ln\left|x+2+\sqrt{x^2+4x-1}\right| + C$

   (b) $4\sqrt{x^2+6x+10} - 10\ln\left|x+3+\sqrt{x^2+6x+10}\right| + C$

   (c) $\ln\sqrt{x^2-2x-5} + C$

5. $1500000\pi[(\pi/8)(\sqrt{4-\pi^2/16}-1) + 2\sin^{-1}(\pi/8) - 1/4] \approx 4.2\times10^6$ kg

10.2 Partial Fractions

PREREQUISITES

1. Recall how to factor a polynomial (Section R.1).
2. Recall how to integrate by the method of trigonometric substitution (Section 10.1).
3. Recall how to integrate by the method of substitution (Section 7.2).

PREREQUISITE QUIZ

1. Evaluate $\int\left(x^2/\sqrt{4 - x^2} + x + 2\right)dx$.
2. Evaluate $\int\left(1/\sqrt{x^2 + 2x + 2}\right)dx$.
3. Factor the following polynomials:
   (a) $x^3 - 27$
   (b) $x^3 - x^2$
   (c) $x^2 + 5x + 6$

GOALS

1. Be able to integrate rational expressions by the technique of partial fractions.

STUDY HINTS

1. Beginning partial fractions. Look before you leap! There may be an easier method. See Example 7. If partial fractions is the method of choice, be sure the degree of the denominator is larger than the degree of the numerator. If not, begin with long division.
2. Denominator factorization. All factors should be of degree one or two. If not, the denominator can be factored further. Check to be sure quadratic factors do not factor further (by using the quadratic formula,

2. (continued)

if necessary). Don't forget that $x = x - 0$, so x^2 is composed of the linear factors $(x - 0)\cdot(x - 0)$. See Example 4. Also, remember that a factor raised to the n^{th} power must be represented n times.

3. Determination of coefficients. This method is called comparing coefficients: Multiply so that both sides of the equation are over a common denominator. Rather than expanding, it is best to leave the expression as a sum of factored terms. Then, substitute values of x so that as many terms as possible become zero. The result should be a few, simple linear equations. If no such x's are left, choose any other constants.

4. Differentiating to determine coefficients. To solve Example 2, this author prefers Method 1. I find that there's a greater chance for error with the differentiation process; you may disagree.

5. Comparing integrands. After you have found the coefficients, use a calculator and an arbitrary number to compare the original integrand with your new one.

6. Rationalizing substitutions. If $[f(x)]^{p/q}$ appears in the integrand, you might be able to make a simplification by substituting $u = [f(x)]^{1/q}$, i.e., $u^q = f(x)$. See Example 8.

7. Integrals of rational trigonometric expressions. The technique used in Examples 10 and 11 is useful for rational functions in $\sin x$ and $\cos x$ Rather than memorizing equations (8), (9), and (10), it is suggested that you reproduce Fig. 10.2.2, and use the identities $\sin 2x = 2 \sin x \cos x$ and $\cos 2x = \cos^2 x - \sin^2 x$ to complete the substitution. Often, partial fractions may be necessary to finish the integration.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Let $1/(x-2)^2(x^2+1)^2 = A/(x-2) + B/(x-2)^2 + (Cx+D)/(x^2+1) + (Ex+F)/(x^2+1)^2$. Then we need to solve $1 = A(x-2)(x^2+1)^2 + B(x^2+1)^2 + (Cx+D)(x-2)^2(x^2+1) + (Ex+F)(x-2)$. If $x = 2$, $1 = B(2^2+1)^2$, i.e., $B = 1/25$. Substituting in $B = 1/25$ and comparing coefficients gives $A + C = 0$; $-2A - 4C + D = -1/25$; $2A + 5C - 4D + E = 0$; $-4A - 4C + 5D - 4E + F = -2/25$; $A + 4C - 4D + 4E - 4F = 0$; and $-2A + 4D + 4F = 24/25$. Solving this set of equations gives: $A = -8/125$, $B = 5/125$, $C = 8/125$, $D = 11/125$, $E = 20/125$, and $F = 15/125$. Thus, $\int dx/[(x-2)^2(x^2+1)^2] = (1/125)\int[-8/(x-2) + 5/(x-2)^2 + (8x+11)/(x^2+1) + (20x+15)/(x^2+1)^2]dx = (1/125) \times \{-8\ln|x-2| - 5/(x-2) + 4\int[2x\,dx/(x^2+1)] + 11\tan^{-1}x + 10\int[2x/(x^2+1)^2]dx + 15\int[dx/(x^2+1)^2]\}$. The last integral can be evaluated by using trigonometric substitution with $\tan\theta = x$, i.e., $\int[dx/(x^2+1)^2] = \int(\sec^2\theta\,d\theta/\sec^4\theta) = \int\cos^2\theta\,d\theta = \int[(1+\cos 2\theta)/2]d\theta = \theta/2 + \sin 2\theta/4 + C = \theta/2 + \sin\theta\cos\theta/2 + C = (1/2)\tan^{-1}x + x/2(1+x^2) + C$. Hence, $\int[dx/(x-2)^2(x^2+1)^2] = (1/125)\{-8\ln|x-2| - 5/(x-2) + 4\ln(x^2+1) + 11\tan^{-1}x - 10/(x^2+1) + (15/2)\tan^{-1}x + 15x/2(1+x^2)\} + C = (1/125)\{4\ln[(x^2+1)/(x^2-4x+4)] + (37/2)\tan^{-1}x + (15x-20)/2(1+x^2) - 5/(x-2)\} + C$.

5. Since the discriminant $(b^2 - 4ac)$ for $x^2 + 2x + 2$ is less than zero, we cannot factor it further. Thus, $x^2/(x-2)(x^2+2x+2) = A/(x-2) + (Bx+C)/(x^2+2x+2) = [(A+B)x^2 + (2A-2B+C)x + (2A-2C)]/(x-2)(x^2+2x+2)$. Comparing coefficients and solving simultaneous equations gives $A = 2/5$, $B = 3/5$, and $C = 2/5$. Thus, $\int[x^2/(x-2)(x^2+2x+2)]dx = (1/5)\int[2/(x-2) + (3x+2)/(x^2+2x+2)]dx$.

5. (continued)

The first term gives $\int[2/(x-2)]\,dx = \ln[(x-2)^2]$. For the second term, note that $(3x+2)/(x^2+2x+2) = (3/2)[(2x+2)/(x^2+2x+2)] - 1/(x^2+2x+1+1)$. The reason we break it up this way is because we want to have $d(x^2+2x+2)$ in the numerator and whatever is left constitutes another term. Thus, $\int[(3x+2)/(x^2+2x+2)]\,dx = (3/2) \times \int[(2x+2)/(x^2+2x+2)]\,dx - \int[1/((x+1)^2+1)]\,dx = (3/2)\ln(x^2+2x+2) - \tan^{-1}(x+1)$. All this information gives $\int[x^2/(x-2)(x^2+2x+2)]\,dx = (1/5)\{\ln(x-2)^2 + (3/2)\ln(x^2+2x+2) - \tan^{-1}(x+1)\} + C$.

9. Factoring x^4+2x^2-3 gives $(x^2+3)(x+1)(x-1)$. Using the technique of partial fractions, we have $x/(x^4+2x^2-3) = (Ax+B)/(x^2+3)$ $C/(x+1) + D/(x-1) = [(Ax+B)(x^2-1) + C(x^2+3)(x-1) + D(x^2+3) \times (x-1)]/(x^2+3)(x^2-1)$. Comparing coefficients gives the following results: x^3: $A+C+D=0$; x^2: $B-C+D=0$; x: $-A+3C+3D=1$ $x^0(1)$: $-B-3C+3D=0$. Solving these four equations simultaneously gives $A=-1/4$, $B=0$, $C=1/8$, and $D=1/8$. Thus, $\int[x/(x^4+2x^2-3)]\,dx = (1/8)\int[-2x/(x^2+3) + 1/(x+1) + 1/(x-1)]\,dx = (1/8) \times [-\ln(x^2+3) + \ln|x+1| + \ln|x-1|] + C = (1/8)\ln|(x^2-1)/(x^2+3)| + C$

13. We apply the method of Example 8 by letting $u=\sqrt{x}$, so $u^2=x$, and $2u\,du = dx$. Thus, $\int[\sqrt{x}/(1+x)]\,dx = \int[u/(1+u^2)]2u\,du = 2\int[u^2/(1+u^2)]\,du = 2\int[(u^2+1-1)/(u^2+1)]\,du = 2\int du - 2\int[1/(u^2+1)]\,du = 2u - 2\tan^{-1}u + C = 2\sqrt{x} - 2\tan^{-1}\sqrt{x} + C$.

17. We use the technique of Example 11. Substitute $u=\tan(x/2)$, so $\sin x = 2u/(1+u^2)$ and $dx = 2du/(1+u^2)$. Thus, $\int[dx/(1+\sin x)] = \int\{[2du/(1+u^2)]/[1+2u/(1+u^2)]\} = \int[2du/(1+u^2+2u)] = 2\int(u+1)^{-2}du = -2(u+1)^{-1} + C = -2/(1+\tan(x/2)) + C$.

21. By the shell method, $V = 2\pi\int_5^6 [x/((1-x)(1-2x))]\,dx$. Now, partial fractions yield $2\pi\int_5^6 [A/(1-x) + B/(1-2x)]\,dx$, with $A/(1-x) + B/(1-2x) = x/((1-x)(1-2x))$. Therefore, $A(1-2x) + B(1-x) = x$ for all x . Let $x = 1/2$, so $B/2 = 1/2$, i.e., $B = 1$. Let $x = 1$, so $-A = 1$, i.e., $A = -1$. Therefore, $V = 2\pi\int_5^6 [-1/(1-x) + 1/(1-2x)]\,dx = 2\pi[\ln(x-1) - (1/2)\ln(2x-1)]\big|_5^6 = 2\pi(\ln 5 - \ln\sqrt{11} - \ln 4 + \ln 3) = \pi\ln(225/176)$.

25. (a) Integrating the right-hand side and using the method of partial fractions for the left-hand side, we get $kt + C = \int [A/(80-x) + B/(60-x)]\,dx$ for constants A , B , and C , where $A(60-x) + B(80-x) = 1$ for all x . Let $x = 60$, so $20B = 1$, or $B = 1/20$. Let $x = 80$, so $-20A = 1$, or $A = -1/20$. Thus, $kt + C = (1/20)\int[-1/(80-x) + 1/(60-x)]\,dx = (1/20)\int[1/(x-80) - 1/(x-60)]\,dx = (1/20)\ln|(x-80)/(x-60)|$. Since $x = 0$ when $t = 0$, we have $C = (1/20)\ln(4/3)$, and the formula becomes $kt + (1/20)\ln(4/3) = (1/20)\ln|(x-80)/(x-60)|$.

(b) Rearrangement of the equation in part (a) gives $20kt + \ln(4/3) = \ln|(x-80)/(x-60)|$ and exponentiation yields $(4/3)\exp(20kt) = |(x-80)/(x-60)|$. Since we assume $x < 60$, the formula becomes $(4/3)\exp(20kt) = (x-80)/(x-60)$ or $(4/3)(x-60) \times \exp(20kt) = x - 80$. Rearrange again to get $[(4/3)\exp(20kt) - 1]x = -80 + 80\exp(20kt)$. Thus, $x = 80[1 - \exp(20kt)]/[1 - (4/3)\exp(20kt)]$

(c) If $x = 20$ when $t = 10$, we substitute into the formula in part (a) to get $(1/20)\ln(3/2) = 10k + (1/20)\ln(4/3)$ or $(1/20)[\ln(3/2) - \ln(4/3)] = (1/20)\ln(9/8) = 10k$. Therefore, $k = \ln(9/8)/200$. Now, substitute $k = \ln(9/8)/200$ and $t = 15$ into the formula in part (b) to get $x = 80[1 - \exp(3\ln(9/8)/2)]/[1 - (4/3)\exp(3\ln(9/8)/2]$ = $80(1 - (9/8)^{3/2})/(1 - (4/3)(9/8)^{3/2}) \approx 26.2$ kg .

SECTION QUIZ

1. Evaluate $\int [dx/(2x^3 + 4x^2)]$.

2. Evaluate $\int_1^{\sqrt{2}} [dx/(x^2 + 2)^2 x]$.

3. Evaluate $\int [dx/(x^2 - 2)(x - 1)]$.

4. Find the average of $f(t) = t^8\sqrt{t^3 + 2}/(1 - t^6)$ on the interval $2 \leqslant t \leqslant 4$. (Hint: let $u = \sqrt{t^3 + 2}$.)

5. Evaluate $\int_0^{\pi/4} [\tan\theta/(1 + \sec\theta)]\,d\theta$.

6. When the minister asked for any objections, the bride's grandfather said, "Sonny-boy has to prove to me he's smart enough to be her husband. Look at that archway. $y = 10 - x/(x^2 - 2x - 3)$ on $[0,2]$ describes it. He can marry her if he knows the area under that archway." What answer would bring happiness to the young couple?

ANSWERS TO PREREQUISITE QUIZ

1. $2\sin^{-1}(x/2) - x\sqrt{4 - x^2}/2 + x^2/2 + 2x + C$

2. $x + 1 + \sqrt{x^2 + 2x + 2} + C$

3. (a) $(x - 3)(x^2 + 3x + 9)$

 (b) $x^2(x - 1)$

 (c) $(x + 3)(x + 2)$

ANSWERS TO SECTION QUIZ

1. $(1/8)\ln|(x + 2)/x| - 1/4x + C$

2. $(1/8)\ln(3/2) - 1/48$

3. $[(2 + \sqrt{2})/4]\ln|x + \sqrt{2}| + [(2 - \sqrt{2})/4]\ln|x - \sqrt{2}| - \ln|x - 1| + C$

4. $-(1/9)(66\sqrt{66} - 10\sqrt{10}) + (1/12)\ln[(\sqrt{66} - 1)(\sqrt{10} + 1)/(\sqrt{66} + 1)(\sqrt{10} - 1)] + (1/4\sqrt{3})\ln[(\sqrt{66} + \sqrt{3})(\sqrt{10} - \sqrt{3})/(\sqrt{66} - \sqrt{3})(\sqrt{10} + \sqrt{3})] \approx -56.14$

5. $\ln[1 + \tan^2(\pi/8)]$

6. $20 - \ln\sqrt{3}$

10.3 Arc Length and Surface Area

PREREQUISITES

1. Recall how to integrate by using trigonometric substitution (Section 10.1).
2. Recall how to compute the distance between two points in the plane (Section R.4).

PREREQUISITE QUIZ

1. What is the distance between the points $(3,6)$ and $(4,2)$?
2. Evaluate $\int (dx/\sqrt{x^2 - 1})$.

GOALS

1. Be able to express the length of a curve as an integral and perform the integration, if possible.
2. Be able to express the area of a surface of revolution as an integral and perform the integration, if possible.

STUDY HINTS

1. Definition. The notation ds is introduced in this section. It is an infinitesimal length defined by $\sqrt{(dx)^2 + (dy)^2}$. See Fig. 10.3.1.
2. Arc length. You should become familiar with $L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$. Either memorize this or learn to derive it. Starting with $ds = \sqrt{(dx)^2 + (dy)^2}$, substitute $dy = f'(x)dx$, factor out dx , and integrate. Since the integrand is positive, you made a mistake if your answer is negative or zero.
3. Constant with no effect. The length of $f + C$ is the same as that of f . Shifting a graph along the y-axis does not change its length. Compute the length of $(x - 1)^{3/2}$ on $[1,2]$ and compare with Example 2.

4. Tricks to find arc length. In general, many textbook and exam questions are chosen for their simplicity. Thus, the expression under the radical will often simplify; for example, it may be a perfect square. If it is not a perfect square, a trigonometric substitution will often be helpful.

5. Square roots in arc length problems. The appearance of square roots in arc length questions may sometimes present a special problem. When in doubt, take the absolute value of the expression inside the square root. For example, consider finding the arc length of $x^{2/3}$ on $[-8,-2]$. We get $\int_{-8}^{-2}\sqrt{(9x^{2/3}+4)/9x^{2/3}}\,dx = \int_{-8}^{-2}\left(\sqrt{9x^{2/3}+4}/3x^{1/3}\right)dx$ and substituting $u = 9x^{2/3}+4$ yields $\int_{\sqrt{40}}^{\sqrt{9(4)^{1/3}+4}}\sqrt{u}\,du/18 = (1/27)u^{3/2}\Big|_{\sqrt{40}}^{\sqrt{9(4)^{1/3}+4}}$, which is negative. The correct method is to use $\sqrt{9x^{2/3}} = 3|x^{1/3}|$.

6. Step function derivation (pp. 480-482). Except in honors classes, most instructors will not emphasize these theoretical aspects on their exams.

7. Area of revolution. Learn to derive formula (2) in the box on p. 483. It may be easier to think of the infinitesimal frustums as cylinders, so the area is circumference times width. The circumference is $2\pi r = 2\pi f(x)$ and the width is simply the arc length. Thus, $A = \int_a^b 2\pi f(x)L\,dx = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$.

8. Revolution around y-axis. Here, the circumference is $2\pi x$ rather than than $2\pi y$. The width is still the arc length. Thus, $A = 2\pi\int_a^b x\sqrt{1+[f'(x)]^2}\,dx$. The only difference with the preceding formula is that $f(x)$ appears in one and x appears in the other.

9. Mathematical illusion? The bands in Fig. 10.3.8 are equal in area. Why? The smaller radius is compensated for by a larger ds .

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Given that $f(x) = x^4/8 + 1/4x^2$, we have $f'(x) = x^3/2 - 1/2x^3$. Thus the length of the graph is $L = \int_1^3\sqrt{1 + (x^3/2 - 1/2x^3)^2}\,dx = \int_1^3\sqrt{x^6/4 + 1/2 + 1/4x^6}\,dx = \int_1^3(x^3/2 + 1/2x^3)dx = (x^4/8 - 1/4x^2)\Big|_1^3 = 92/9$.

5. If $f(x) = x^n$, then $f'(x) = nx^{n-1}$ and $\sqrt{1 + [f'(x)]^2} = \sqrt{1 + n^2x^{2n-2}}$. Thus, the length is $L = \int_a^b\sqrt{1 + n^2x^{2n-2}}\,dx$.

9. This exercise is analogous to Example 5. By direct computation, we have $L = [(1 - 0)^2 + (2 - 0)^2]^{1/2} + [(2 - 1)^2 + (1 - 2)^2]^{1/2} + [(5 - 2)^2 + (0 - 1)^2]^{1/2} = \sqrt{5} + \sqrt{2} + \sqrt{10}$. On the other hand, $f'(x) = 2$ on $[0,1]$, -1 on $[1,2]$, and $-1/3$ on $[2,5]$. Thus, the length is $\int_0^1\sqrt{1 + (2)^2}\,dx + \int_1^2\sqrt{1 + (-1)^2}\,dx + \int_2^5\sqrt{1 + (-1/3)^2}\,dx = \sqrt{5}x\Big|_0^1 + \sqrt{2}x\Big|_1^2 + \sqrt{10/9}x\Big|_2^5 = \sqrt{5} + \sqrt{2} + \sqrt{10}$.

13. Given that $f(x) = (x + 1)^{1/2}$, we have $f'(x) = (1/2)(x + 1)^{-1/2}$. Thus, the area of the surface obtained by revolving $f(x)$ on $[0,2]$ about the x-axis is $A = 2\pi\int_0^2(x + 1)^{1/2}\sqrt{1 + [(1/2)(x + 1)^{-1/2}]^2}\,dx = 2\pi\int_0^2[\sqrt{4x + 5}/2]\,dx = (\pi/6)(4x + 5)^{3/2}\Big|_0^2 = (\pi/6)(13^{3/2} - 5^{3/2}) \approx 18.7$.

17. $y' = -\sin x$, so the area is $A = 2\pi\int_{-\pi/2}^{\pi/2}\cos x\sqrt{1 + \sin^2 x}\,dx$. Let $u = \sin x$ so $du = \cos x\,dx$. Then $A = 2\pi\int_{-1}^{1}\sqrt{1 + u^2}\,du$. Integrate as in Example 3 to get $A = 2\pi[(u/2)\sqrt{u^2 + 1} + (1/2)\ln|u + \sqrt{u^2 + 1}|]\Big|_{-1}^{1} = (2\pi)[(1/2)\sqrt{2} + (1/2)\ln(1 + \sqrt{2}) + (1/2)\sqrt{2} - (1/2)\ln(\sqrt{2} - 1)] = [\sqrt{2} + \ln\sqrt{(1 + \sqrt{2})/(\sqrt{2} - 1)}]2\pi = 2\pi(\sqrt{2} + \ln(1 + \sqrt{2})) \approx 14.42$.

21. This exercise is analogous to Example 9. We have $f(x) = x$ on $[0,1]$ and $f(x) = -x + 2$ on $[1,2]$. Thus, the area is $A = 2\pi\left[\int_0^1 x\sqrt{1 + 1^2}\,dx + \int_1^2(-x + 2)\sqrt{1 + (-1)^2}\,dx\right] = 2\sqrt{2}\pi[\int_0^1 x\,dx + \int_1^2(-x + 2)dx] = 2\sqrt{2}\pi[(x^2/2)\Big|_0^1 + (-x^2/2 + 2x)\Big|_1^2] = 2\sqrt{2}\pi[(1/2) + (-3/2 + 2)] = 2\sqrt{2}\pi$.

25. $[f'(x)]^2 = (3a\sqrt{x+b}/2)^2 = 9a^2(x+b)/4$, so the length $= s = \int_0^1\sqrt{1+9a^2(x+b)/4}\,dx$. Let $u = 1+9a^2(x+b)/4$, so $du = 9a^2dx/4$; i.e., $dx = 4\,du/9a^2$. Then $s = \int_{x=0}^{x=1} 4u^{1/2}du/9a^2 = (8/27a^2)(1+9a^2(x+b)/4)^{3/2}|_0^1 = (8/27a^2)[(1+9a^2(1+b)/4)^{3/2} - (1+9a^2b/4)^{3/2}]$. Note that $4^{3/2} = 8$, so $s = (1/27a^2)[(4+9a^2(1+b))^{3/2} - (4+9a^2b)^{3/2}]$. Changing c has no effect, since c dropped out when we found $f'(x)$.

29. $f'(x) = \sec^2 x + 2$, so the length is $\int_0^{\pi/2}\sqrt{1+[f'(x)]^2}dx = \int_0^{\pi/2}\sqrt{5+\sec^4 x+4\sec^2 x}\,dx$. And the area is $2\pi\int_0^{\pi/2}f(x)\sqrt{1+[f'(x)]^2}dx = 2\pi\int_0^{\pi/2}(\tan x+2x)\sqrt{5+\sec^4 x+4\sec^2 x}\,dx$.

33. The area of a surface of revolution is $2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}dx = 2\pi\int_a^b f(x)ds$. For a small arc length where $f(x)$ is almost constant, this is approximately 2π multiplied by the function value multiplied by the small length. To find the area of the surface obtained by revolving the given curve, divide the curve into 1 mm segments. On each segment, measure the distance from the x-axis to each endpoint, and then take the average value as $f(x)$. Using this method, our answer is about $16\ \text{cm}^2$.

37. Let s_1 be the length of $\sin x$ and s_2 be the length of $1+x^4$. Then $s_1 = \int_{0.1}^1\sqrt{1+\cos^2 x}\,dx$. Since $\cos x \leqslant 1$, $s_1 \leqslant \int_{0.1}^1\sqrt{2}\,dx = \sqrt{2}(1-0.1) = 0.9\sqrt{2} \approx 1.28$. Now s_2 must be longer than the line from $(0.1, 1.0001)$ to $(1,2)$, so $s_2 > \sqrt{(0.9)^2+(0.9999)^2} \approx 1.34$. Thus, $s_2 > s_1$.

41. (a) Cutting and unrolling the frustum as in Fig. 10.3.12, we found the area to be $\pi s(r_1+r_2)$ if a linear function is revolved around the x-axis. Since the formula is independent of x and y , the formula for the area obtained by revolving a linear segment around

41\. (a) (continued)

the y-axis is also $\pi s(r_1 + r_2)$, which works out to be $\pi(a + b)\sqrt{1 + m^2}(b - a)$, since $s = \sqrt{1 + m^2}(b - a)$.

(b) From part (a), the area is $\pi\sqrt{1 + m^2}(b^2 - a^2) = 2\pi\sqrt{1 + m^2}(b^2/2 - a^2/2) = 2\pi\sqrt{1 + m^2}(x^2/2)\Big|_a^b = \int_a^b 2\pi x\sqrt{1 + m^2}dx$. Since $m = f'(x)$, we have $A = 2\pi\int_a^b x\sqrt{1 + [f'(x)]^2}dx$. Let the entire surface be composed of a finite sequence of frustums of cones and use the additivity property of the integral to obtain formula (3).

SECTION QUIZ

1\. Compute the arc length of the following functions:

(a) $y = x^5/2 + 1/30x^3$ on $[1,t]$, $t > 1$

(b) $y = \sqrt{x}$ on $[1,2]$ (Hint: Substitute $u = \sqrt{4x}$.)

(c) $y = (2x + 5)^{3/2} + 3$ on $[0,3]$

2\. Suppose the arc length of f on $[a,b]$ is L . What can you say about the arc length of $g = f + k$ on $[a,b]$, where k is a constant?

3\. Find the area of the surfaces obtained by revolving the curves in Question 1 around the y-axis. (Hint: Use formulas 65 and 66 of the integral table for part (b).)

4\. Find the area of the surfaces obtained by revolving the curves in Question 1, parts (a) and (b), around the x-axis.

5\. Upon close inspection, you notice that your pet tarantula's legs can be described by $y = 1 - t^{3/2}$ on $[0,1]$.

(a) How long are the tarantula's legs?

One day, while you were walking your tarantula, a cyclone suddenly appeared and twirled the spider around so fast that it seemed like a

5. (continued)

solid surface was being formed by revolving one of the legs around the y-axis.

(b) What was the apparent surface area?

ANSWERS TO PREREQUISITE QUIZ

1. $\sqrt{17}$

2. $\ln|x + \sqrt{x^2 + 1}| + C$

ANSWERS TO SECTION QUIZ

1. (a) $t^5/2 + 1/30t^3 - 7/15$

   (b) $(1/4)[6\sqrt{2} - 2\sqrt{5} + \ln((\sqrt{8} + 3)/(\sqrt{5} + 2))]$

   (c) $(1000 - 46\sqrt{46})/27$

2. They are equal.

3. (a) $\pi(5t^6/6 - 1/10t^2 - 11/15)$

   (b) $(51\sqrt{2} - 9\sqrt{5})/16 - (1/64)\ln(17 + 12\sqrt{12})/(9 + 4\sqrt{5}))$

   (c) $2\pi(35000 - 2116\sqrt{46})/1215$

4. (a) $\pi(t^{10}/4 - t^2/30 + 1/900t^6 - 49/225)$

   (b) $(\pi/6)(27 - 5\sqrt{5})$

5. (a) $(13\sqrt{13} - 8)/27$

   (b) $16(884\sqrt{13} - 4)/1215$

10.4 Parametric Curves

PREREQUISITES

1. Recall how to sketch curves described by parametric equations (Section 2.4).
2. Recall how to compute arc lengths (Section 10.3).

PREREQUISITE QUIZ

1. Sketch the curve described by $x = 2t + 1$ and $y = t + 2$.
2. Sketch the curve described by $x = t^2$ and $y = t^4$.
3. Compute the length of the graph of $y = 2x + 3$ on $[0,1]$.
4. Write a formula for the length of the graph of $y = x^3 - x^2$ on $[1,2]$. (Do not evaluate.)

GOALS

1. Be able to find the tangent line of a parametric curve.
2. Be able to relate speed to arc length for parametric curves.
3. Be able to express the length of a parametric curve as an integral and perform the integration, if possible.

STUDY HINTS

1. Direction of motion. Parametric curves move in a specified direction. This is illustrated in Example 1.
2. Converting parametric equations. One of the simplest ways to convert a parametric equation into the form $y = f(x)$ is to solve one equation for t and then substitute into the other equation.
3. Tangent lines. You should definitely know how to compute a tangent line for parametric equations. Remember that for $y = f(x)$, the tangent

3. (continued)

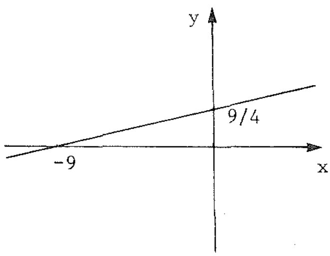
line is given by $y = y_0 + f'(x_0)(x - x_0)$. For the analogous parametric form, $f'(x_0)$ is replaced by $g'(t_0)/f'(t_0)$, y_0 by $g(t_0)$, and x_0 by $f(t_0)$.

4. Arc length. The infinitesimal argument is very easy to follow. Starting with $ds = \sqrt{dx^2 + dy^2}$, we multiply and divide by dt to get $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$. Integration yields the arc length formula.

5. Speed. It is probably best to memorize that speed is given by the formula $\sqrt{(dx/dt)^2 + (dy/dt)^2}$, which is the arc length integrand.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1.



From $x = 4t - 1$, we get $t = (x + 1)/4$, and so $y = t + 2 = (x + 1)/4 + 2 = (1/4) \times (x + 9)$. The graph is a straight line with slope $1/4$ and y-intercept $9/4$.

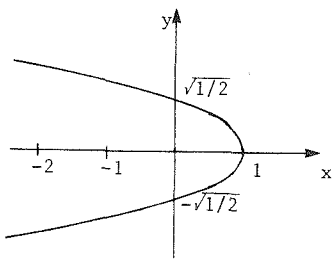
5. One of the simplest parametric representations is to let $x = t$. $2x^2 + y^2 = 1$ becomes $y = \pm\sqrt{1 - 2x^2}$, so a parametric representation is $x = t$ and $y = \pm\sqrt{1 - 2t^2}$. Another one is $x = \sin t/\sqrt{2}$ and $y = \cos t$. Many other solutions are possible.

9. If we let $x = t$, then $y = t^3 + 1$. We can also let $x = \pm t^2$, so $y = \pm t^6 + 1$. Several other solutions are possible.

13. Given $x = f(t) = (1/2)t^2 + t$ and $y = g(t) = t^{2/3}$, we have $f'(t) = t + 1$ and $g'(t) = (2/3)t^{1/3}$. At $t_0 = 1$, we have $f(1) = 3/2$, $g(1) = 1$, $f'(1) = 2$, and $g'(1) = 2/3$. So the tangent line has the equation $y = [(2/3)/2](x - 3/2) + 1$, or $y = (1/3)(x + 3/2)$.

17. The parametric form of the tangent line equation is: $x = f'(t_0) \times (t - t_0) + f(t_0)$ and $y = g'(t_0)(t - t_0) + g(t_0)$. Here, $f'(t) = 2(2 - 3t)(-3)$ and $g'(t) = -3$. Thus, at $t_0 = 1$, we have $f(1) = 1$, $f'(1) = 6$, $g(1) = -1$ and $g'(1) = -3$; therefore, $x = 6(t - 1) + 1$ and $y = -3(t - 1) - 1$. At $t = 3$, the bead is at $(13,-7)$.

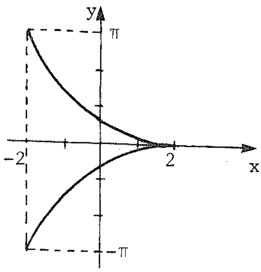
21.



Since $\sin t = \pm\sqrt{(1 - \cos 2t)/2}$, $y = \pm\sqrt{(1 - x)/2}$, i.e., $y^2 = (1 - x)/2$. $x'(t) = -2 \sin 2t$ and $y'(t) = \cos t$. The tangent line should be horizontal when $y' = 0$, i.e., $t = (2n + 1)\pi/2$, where n is an integer. However, $x' = 0$ also at these points. From the graph, we see that the tangent is never horizontal. The graph should be vertical at $2t = n\pi$, i.e., $t = n\pi/2$. If n is even, $y' \neq 0$, so there is a vertical tangent. If n is odd, $y' = 0$, and the limit of y'/x' shows that the tangent is not vertical. So vertical tangents occur when $t = m\pi$ for all integers m .

25. Since $dx/dt = 2t$ and $dy/dt = 4t^3$, the length is $s = \int_0^1 \sqrt{4t^2 + 16t^6}\, dt = \int_0^1 2t\sqrt{1 + 4t^4}\, dt$. Let $u = 2t^2$ so $du = 4t\, dt$. Then $s = (1/2) \times \int_0^2 \sqrt{1 + u^2}\, du$. Integrate as in Example 3, Section 10.3, to get $s = (1/2) \times [(u/2)\sqrt{u^2 + 1} + (1/2) \ln(u + \sqrt{u^2 + 1})] \big|_0^2 = (1/2)[\sqrt{5} + (1/2) \ln(2 + \sqrt{5})] \approx 1.48$.

29. (a) $dx/d\theta = -2 \sin \theta$ and $dy/d\theta = 1 - \cos \theta$. Then $dy/dx = (1 - \cos \theta)/(-2 \sin \theta)$. When $\theta = \pi/2$, $dy/dx = 1/(-2) = -1/2$, $x = 0$, and $y = \pi/2 - 1$. The tangent line is $[y - (\pi/2 - 1)]/x = -1/2$, so $y = -x/2 + \pi/2 - 1$.

29\. (b) 

$\theta = \cos^{-1}(x/2)$ and $\sin\theta = \sqrt{1 - x^2/4}$,
so plot $y = \cos^{-1}(x/2) - \sqrt{1 - x^2/4}$.

(c) Let s denote the length. $ds = \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\,d\theta$, so
$s = \int_0^\pi \sqrt{4\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta}\,d\theta =$
$\int_0^\pi \sqrt{5 - 3\cos^2\theta - 2\cos\theta}\,d\theta$.

33\. (a) $\dot{x} = k(\cos\omega t - \omega t\sin\omega t)$; $\dot{y} = k(\sin\omega t + \omega t\cos\omega t)$.

(b) The speed is $\sqrt{\dot{x}^2 + \dot{y}^2} = k(\cos^2\omega t - 2\omega t\sin\omega t\cos\omega t +$
$\omega^2 t^2\sin^2\omega t + \sin^2\omega t + 2\omega t\sin\omega t\cos\omega t + \omega^2 t^2\cos^2\omega t)^{1/2} =$
$k\sqrt{1 + \omega^2 t^2}$.

(c) $\ddot{x} = k(-\omega\sin\omega t - \omega\sin\omega t - \omega^2 t\cos\omega t) = -k\omega(2\sin\omega t +$
$\omega t\cos\omega t)$, and $\ddot{y} = k(\omega\cos\omega t + \omega\cos\omega t - \omega^2 t\sin\omega t) =$
$k\omega(2\cos\omega t - \omega t\sin\omega t)$. Hence, $\ddot{x}(0) = 0$ and $\ddot{y}(0) = 2k\omega$,
so the Coriolis force is $m\sqrt{(2k\omega)^2} = 2mk\omega$.

37\. (a) Using string on the map of The United States in the National Geographic Atlas of the World, p. 25, (scale 1:7,800,000), the coastline of Maine was estimated at 338 miles.

(b) Using string on the map of Maine in the State Farm Road Atlas, Rand McNally, 1974, p. 47, (scale 1 in. = 20 mi) gives 688 miles.

(c) It would probably be longer.

(d) The measurement will depend on the definition and on the scale (and therefore the available detail) of the maps used.

(e) From the World Almanac and Book of Facts - 1974, Newspaper Enterprise Assoc., New York, 1973, p. 744, we have coastline: 228 miles; shoreline: 3,478 miles.

SECTION QUIZ

1. (a) Sketch the curve $x = t^2$, $y = t^3$.

   (b) What is the tangent line when $t = 0$?

2. A curve is described by $x = t^5 + 3t^2 - t + 7$ and $y = 3t^4 - 2t^3/3 + t - 4$. What is the equation of the tangent line when $t = 1$?

3. What is the length of the curve in Question 1 for $-1 \leqslant t \leqslant 1$?

4. What is the speed of a particle moving according to the parametric equations given in Question 2 when $t = 0$?

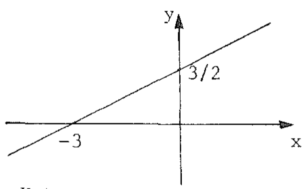
5. The human cannonball's path can be parametrically described by $x = 5t$ and $y = 5\sqrt{3}t - 5t^2$. She is shot through a ring of fire at the highest point of her flight and ends her feat by diving into a narrow tube of water at $y = 0$. The cannon is located at $(0,0)$.

   (a) How fast is she going through the ring of fire?

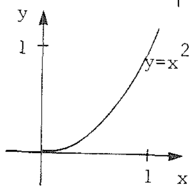
   (b) At what speed and at what angle does she enter the water?

ANSWERS TO PREREQUISITE QUIZ

1.



2.

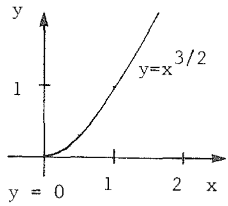


3. $\sqrt{5}$

4. $L = \int_1^2 \sqrt{1 + (3x^2 - 2x)^2}\,dx$

ANSWERS TO SECTION QUIZ

1. (a)



(b) $y = 0$

2. $30y - 33x + 350 = 0$

3. $2(13^{3/2} - 8)/27$

4. $\sqrt{2}$

5. (a) 5

(b) 10 ; $-\pi/3$ radians from ground

10.5 Length and Area in Polar Coordinates

PREREQUISITES

1. Recall how to compute arc lengths of parametric curves (Section 10.4).
2. Recall how to convert from cartesian to polar coordinates (Section 5.1).
3. Recall how to sketch graphs described in polar coordinates (Section 5.6).
4. Recall the trigonometric half-angle formulas for sine and cosine (Section 5.1 and inside front cover).

PREREQUISITE QUIZ

1. Which of the following is equivalent to $\sin^2\theta$?

   (a) $2 \sin \theta \cos \theta$

   (b) $(1 + \cos 2\theta)/2$

   (c) $(1 - \cos 2\theta)/2$

2. Sketch the graph of $r = \sin \theta$ in the xy-plane.
3. What is the length of the curve described by $x = 2 \cos \theta$, $y = \sin \theta$ for $0 \leqslant \theta \leqslant \pi$? (Write a formula, but don't evaluate it.)
4. Convert the cartesian coordinates $(3,3)$ to polar coordinates.

GOALS

1. Be able to compute the arc lengths of curves described in polar coordinates.
2. Be able to compute the area of a region described by polar coordinates.

STUDY HINTS

1. Arc length. The derivation of formula (1) on p. 500 is not difficult. Simply use $x = r \cos \theta = f(\theta)\cos \theta$ and $y = r \sin \theta = f(\theta)\sin \theta$, differentiate by the product rule and substitute into $L =$

1. (continued)
$\int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$. Don't forget to change $[a,b]$ to $[\alpha,\beta]$. You may find it easier just to memorize $L = \int_\beta^\alpha \sqrt{(r')^2 + r^2}\, d\theta$.

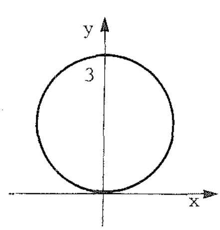
2. Area. You should remember that $A = (1/2)\int_\alpha^\beta r^2 d\theta$. A sketch is usually helpful and noting symmetries will save much work.

3. Choosing limits. In many instances, you will have to find the limits of integration. For arc length problems, be careful the limits are chosen so that the curve is traversed only once. For example, the length of the circle $r = 1$ should be found by integrating from 0 to 2π , not 0 to 4π . For most problems about area, as in Example 3, you will need to find where $f(\theta) = 0$.

4. Area between curves. This is done by computing the larger area and then subtracting the smaller area. Thus, the integrand is $[f(\theta)]^2 - [g(\theta)]^2$, not $[f(\theta) - g(\theta)]^2$.

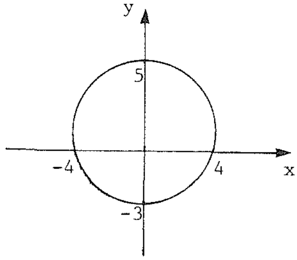
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. We have $dr/d\theta = 3 \cos \theta$, so the length of L is $\int_0^{2\pi} \sqrt{9 \cos^2\theta + 9(1 + \sin \theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{18 + 18 \sin \theta}\, d\theta = \sqrt{18} \times \int_0^{2\pi} \sqrt{1 + \cos(\theta - \pi/2)}\, d\theta$ by the identity $\sin \theta = \cos(\theta - \pi/2)$. By the half-angle formula, this becomes $\sqrt{18}\int_0^{2\pi} \sqrt{2} \cos[(1/2)(\theta - \pi/2)]\, d\theta = 12 \sin[(\theta - \pi/2)/2]\big|_0^{\pi/2} = 12\sqrt{2}$.

5. 

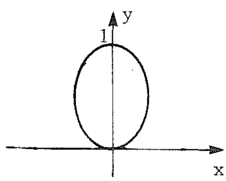
Using the formula $A = (1/2)\int_\alpha^\beta r^2 d\theta$, we get $(1/2)\int_0^\pi (3 \sin \theta)^2 d\theta = (9/2)\int_0^\pi [(1 - \cos 2\theta)/2]\, d\theta = (9/2)(\theta/2 - \sin 2\theta/4)\big|_0^\pi = 9\pi/4$.

9.



The area is $A = (1/2)\int_0^{2\pi}(4 + \sin\theta)^2 d\theta = (1/2)\int_0^{2\pi}(16 + 8\sin\theta + \sin^2\theta)d\theta$. Using the half-angle formula yields $(1/2)\int_0^{2\pi}[16 + 8\sin\theta + (1 - \cos 2\theta)/2]d\theta = (1/2) \times [33\theta/2 - 8\cos\theta - \sin 2\theta/4]\Big|_0^{2\pi} = 33\pi/2$.

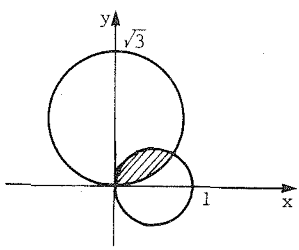
13.



The length is $L = \int_\alpha^\beta \sqrt{r^2 + (r')^2}\, d\theta = \int_{-\pi/2}^{\pi/2}\sqrt{\tan^2(\theta/2) + \sec^2(\theta/2)/4}\, d\theta$. The area is $A = (1/2)\int_\alpha^\beta r^2 d\theta = (1/2)\int_{-\pi/2}^{\pi/2}\tan^2(\theta/2)d\theta = (1/2)\int_{-\pi/2}^{\pi/2}(\sec^2(\theta/2) - 1)d\theta = (\tan(\theta/2) - \theta/2)\Big|_{-\pi/2}^{\pi/2} = 1 - \pi/4 + 1 - \pi/4 = 2 - \pi/2$.

17. The length is $L = \int_0^{\pi/2}\sqrt{(1 + \cos\theta - \theta\sin\theta)^2 + \theta^2(1 + 2\cos\theta + \cos^2\theta)}\, d\theta$. The area is $A = (1/2)\int_0^{\pi/2}(\theta^2 + 2\theta^2\cos\theta + \theta^2\cos^2\theta)d\theta = (1/2)[(\theta^3/3)\Big|_0^{\pi/2} + \int_0^{\pi/2}(2\theta^2\cos\theta + (\theta^2/2)(1 + \cos 2\theta))d\theta] = (1/2)[(\theta^3/2)\Big|_0^{\pi/2} + \int_0^{\pi/2}(2\theta^2\cos\theta + \theta^2\cos 2\theta/2)d\theta]$. Integrate each term by parts to get $A = (1/2)[\pi^3/16 + (2\theta^2\sin\theta + 4\theta\cos\theta - 4\sin\theta + (1/4)\theta^2\sin 2\theta + (1/4)\theta\cos 2\theta - (1/4)\sin 2\theta)]\Big|_0^{\pi/2} = (1/2)[\pi^3/16 + \pi^2/2 - 4 - \pi/8]$.

21.



Since $\cos\theta = \sqrt{3}\sin\theta$ when $\theta = \pi/6$, the length is $L = \int_{\pi/6}^{\pi/2}\sqrt{\sin^2\theta + \cos^2\theta}\, d\theta + \int_0^{\pi/6}\sqrt{3\cos^2\theta + 3\sin^2\theta}\, d\theta = \pi/2 - \pi/6 + \sqrt{3}\pi/6 = (2 + \sqrt{3})\pi/6$. The area is $A = (1/2)[\int_0^{\pi/6}3\sin^2\theta\, d\theta + \int_{\pi/6}^{\pi/2}\cos^2\theta\, d\theta] = (1/2)[(3/2)\int_0^{\pi/6}(1 - \cos 2\theta)d\theta + (1/2) \times \int_{\pi/6}^{\pi/2}(1 + \cos 2\theta)d\theta] = (1/4)[(3\theta - (3/2)\sin 2\theta)\Big|_0^{\pi/6} + (\theta + (1/2) \times \sin 2\theta)\Big|_{\pi/6}^{\pi/2}] = (1/4)[\pi/2 - 3\sqrt{3}/4 + \pi/2 - \pi/6 - \sqrt{3}/4] = (1/4)(5\pi/6 - \sqrt{3})$.

25. Substituting into the formula $L = \int_{\alpha}^{\beta}\sqrt{f(\theta) + [f'(\theta)]^2}\,d\theta$, we get $\int_{2n\pi}^{2(n+1)\pi}\sqrt{e^{2\theta} + e^{2\theta}}\,d\theta = \sqrt{2}\int_{2n\pi}^{2(n+1)\pi} e^{\theta}d\theta = \sqrt{2}[\exp(2(n+1)\pi) - \exp(2n\pi)]$.

SECTION QUIZ

1. (a) Find the area bounded by the curve described by $r = \theta\cos\theta$ and the rays $\theta = 2\pi$ and $\theta = 3\pi$.

   (b) Write the length of the curve as an integral, but do not evaluate it.

2. Find the area and the perimeter of the region enclosed by each of the following:

   (a) $r = \sin\theta + \cos\theta$

   (b) $r = |2\cos\theta|$

3. The region inside both $r = \sin\theta$ and $r = \cos(\theta + \pi/3)$ has area $A = (1/2)[\int_a^b \cos^2(\theta + \pi/3)d\theta + \int_c^d \sin^2\theta\, d\theta]$. Find the limits of integration and compute the area.

4. An astropirate on the surface of Mars was about to accelerate into hyperspace when he noticed that the space patrol seemed confused and was flying circles around him. In reality, the space patrol was relesing a fence described by $r = 1$ and $r = 1 + \cos\theta$.

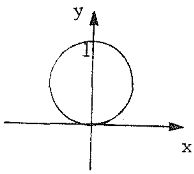
   (a) If the fence has height 1, how much fence material was used? (Assume that the space patrol may penetrate their own fences.)

   (b) The astropirate was caught in one of the regions inside $r = 1$ and outside $r = 1 + \cos\theta$. How much material was needed to make the top and bottom of the space prison?

ANSWERS TO PREREQUISITE QUIZ

1. c

2.



3. $L = \int_0^{\pi} \sqrt{4 \sin^2\theta + \cos^2\theta}\, d\theta$

4. $(3\sqrt{2}, \pi/4)$

ANSWERS TO SECTION QUIZ

1. (a) $19\pi^3/12 + \pi/8$

   (b) $\int_{2\pi}^{3\pi} \sqrt{\theta^2 + \cos^2\theta - \theta \sin 2\theta}\, d\theta$

2. (a) Area = π ; perimeter = $2\sqrt{2}\pi$

   (b) Area = $\pi/2$; perimeter = 2π

3. $a = \pi/12$; $b = \pi/6$; $c = 0$; $d = \pi/12$; area = $\pi/12 - 1/4$

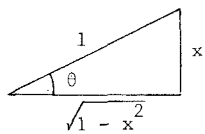
4. (a) $8 + 2\pi$

   (b) $\pi/2 + 4$

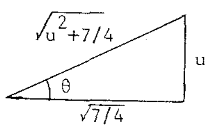
10.R Review Exercises for Chapter 10

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Substitute $u = \sin x$, so $du = \cos x\,dx$, and $\int 3\sin^2 x \cos x\,dx = \int 3u^2 du = u^3 + C = \sin^3 x + C$.

5. 

Let $x = \sin\theta$, so $dx = \cos\theta\,d\theta$ and $\sqrt{1 - x^2} = \cos\theta$. Thus, $\int\left(x^3/\sqrt{1 - x^2}\right)dx = \int(\sin^3\theta/\cos\theta)\cos\theta\,d\theta = \int\sin^3\theta\,d\theta = \int(1 - \cos^2\theta)\sin\theta\,d\theta$. Now, let $u = \cos\theta$, so $-du = \sin\theta\,d\theta$, and we get $-\int(1 - u^2)du = -u + u^3/3 + C = \cos^3\theta/3 - \cos\theta + C$. Using the figure, the answer becomes $(1 - x^2)^{3/2} - \sqrt{1 - x^2} + C$.

9. 

Let $u = x + 1/2$, so $x^2 + x + 2 = (x + 1/2)^2 + 7/4 = u^2 + 7/4$. Now, let $u = \sqrt{7/4}\tan\theta$, so $du = \sqrt{7/4}\sec^2\theta\,d\theta$ and $\sqrt{u^2 + 7/4} = \sqrt{7/4}\sec\theta$. Thus, $\int[dx/(x^2 + x + 2)] = \int[du/(u^2 + 7/4)] = \int[\sqrt{7/4}\sec^2\theta\,d\theta/(7/4)\sec^2\theta] = \sqrt{4/7}\int d\theta = \sqrt{4/7}\,\theta + C = \sqrt{4/7}\tan^{-1}(u/\sqrt{7/4}) + C = (2\sqrt{7}/7)\tan^{-1}[(2x + 1)/\sqrt{7}\,] + C$.

13. Let $u = x^2 + 1$, so $x^2 = u - 1$, and $du = 2x\,dx$. Thus $\int[x^3/(x^2 + 1)^2]dx = \int[(u - 1)du/2u^2] = (1/2)\int(1/u - 1/u^2)du = (1/2)[\ln|u| + 1/u] + C = (1/2)[\ln|x^2 + 1| + 1/(x^2 + 1)] + C$.

17. Let $u = \sqrt{x}$, so $u^2 = x$ and $2u\,du = dx$. Then $\int\sin\sqrt{x}\,dx = 2\int u\sin u\,du$. Integrate by parts to get $-2u\cos u + 2\sin u + C = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$.

21. Using $\sin^2 x = 1 - \cos^2 x$, we get $\int(\sin^2 x/\cos x)dx = \int(\sec x - \cos x)dx = \ln|\sec x + \tan x| - \sin x + C$. The technique for integrating $\sec x$ is shown in Example 6(b), Section 10.1.

25. Let $I = \int [x/(x^3 - 9)]\,dx$ and let $a = \sqrt[3]{9}$. Then $I = \int [x/((x - a)\times (x^2 + ax + a^2))]\,dx = \int [B/(x - a) + (Cx + D)/(x^2 + ax + a^2)]\,dx$, where $B(x^2 + ax + a^2) + (Cx + D)(x - a) = x = (B + C)x^2 + (a(B - C) + D)x - a(aB - D)$. Equating coefficients, we get $B + C = 0$; $a(B - C) + D = 1$; and $aB - D = 0$. Substitute $C = -B$ and $D = aB$ into $a(B - C) + D = 1 = a(2B) + aB = 1$, so $B = 1/3a$. Therefore, $C = -1/3a$ and $D = 1/3$. Thus, $I = (1/3)\int [(1/a)/(x - a) + (-x/a + 1)/(x^2 + ax + a^2)]\,dx = (1/3a)\{\ln|x - a| - (1/2)\int [(2x + a - 3a)/(x^2 + ax + a^2)]\,dx\} = (1/3a)\times \{\ln|x - a| - (1/2)[\ln(x^2 + ax + a^2) - \int (3a/(x + a/2)^2 + 3a^2/4)dx]\} = (1/3a)\{\ln|x - a| - (1/2)[\ln(x^2 + ax + a^2) - (3a)(2/a\sqrt{3})\tan^{-1}[2(x + a/2)/(a\sqrt{3})]]\} + C = (1/3a)[\ln|x - a| - \ln\sqrt{x^2 + ax + a^2} + \sqrt{3}\tan^{-1}((2x/a + 1)/\sqrt{3})] + C = (1/3\sqrt[3]{9})[\ln|x - \sqrt[3]{9}| - \ln\sqrt{x^2 + \sqrt[3]{9}\,x + 3\sqrt[3]{3}} + \sqrt{3}\tan^{-1}((2x/\sqrt[3]{9} + 1)/\sqrt{3})] + C$.

29. $\int (1 + e^x)^{-1}dx = \int [1 - e^x(e^x + 1)^{-1}]\,dx = x - \ln(e^x + 1) + C$, by substituting $u = e^x + 1$.

33. Use $\sin x \cos y = (1/2)[\sin(x + y) + \sin(x - y)]$ to get $\int \sin 3x \cos 2x\,dx = (1/2)\int (\sin 5x + \sin x)dx = -(1/10)\cos 5x - (1/2)\times \cos x + C$.

37. Let $u = \sqrt{x}$, so $du = (1/2\sqrt{x})dx$. Then $\int (e^{\sqrt{x}}/\sqrt{x})dx = 2\int e^u du = 2e^u + C = 2e^{\sqrt{x}} + C$.

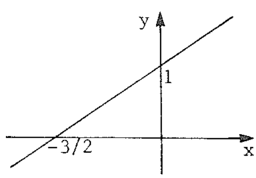
41. Let $u = x^2 + 3$, so $du = 2x\,dx$. Then $\int [x/(x^2 + 3)]\,dx = (1/2)\int (du/u) = (1/2)\ln|u| + C = (1/2)\ln(x^2 + 3) + C = \ln\sqrt{x^2 + 3} + C = (1/2)\ln(x^2 + 3) + C$.

45. Let $u = (\ln 3x) + 5$, so $du = (3/3x)dx = dx/x$. Then $\int_1^2 [(\ln 3x + 5)^3/x]\,dx = \int_{\ln 3+5}^{\ln 6+5} u^3 du = (u^4/4)\Big|_{\ln 3+5}^{\ln 6+5} = (1/4)[(\ln 6 + 5)^4 - (\ln 3 + 5)^4] \approx 186.1$.

49. Let $u = \cos\theta$, so $-du = \sin\theta\,d\theta$. Then $\int_0^{2\pi} [\sin\theta/(1 + \cos\theta + \cos^2\theta)]\,d\theta = -\int_1^1 [1/(1 + u + u^2)]\,du$. Since the limits of integration are both 1, the integral is 0.

53. Use $L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$, where $f'(x) = x^2 - 1/4x^2$. Thus, $L = \int_1^2 \sqrt{1 + (x^2 - 1/4x^2)^2}\,dx = \int_1^2 \sqrt{1 + (x^4 - 1/2 + 1/16x^4)}\,dx = \int_1^2 (x^2 + 1/4x^2)\,dx = (x^3/3 - 1/4x)\big|_1^2 = 7/3 - (1/8 - 1/4) = 59/24$.

57. Since the graph is revolved around the y-axis, use the equation $A = 2\pi\int_a^b x\sqrt{1 + [f'(x)]^2}\,dx$. Here, $f'(x) = 1/x \ln 10$, so $A = 2\pi\int_{10}^{100} x\sqrt{1 + (1/x \ln 10)^2}\,dx = 2\pi\int_{10}^{100}\sqrt{x^2 + 1/(\ln 10)^2}\,dx$. Integrate as in Example 3, Section 10.3 to get $A = 2\pi[(x/2)\sqrt{x^2 + 1/(\ln 10)^2} + (1/2(\ln 10)^2)\ln|x + \sqrt{x^2 + 1/(\ln 10)^2}|]\,\big|_{10}^{100} = 2\pi[50\sqrt{10000 + 1/(\ln 10)^2} - 5\sqrt{100 + 1/(\ln 10)^2} + (1/2(\ln 10)^2)\ \ln[(100 + \sqrt{10000 + 1/(\ln 10)^2})/(10 + \sqrt{100 + 1/(\ln 10)^2})]\ \approx 31103$.

61. 

Solving for t , we get $t = x/3$. Substitute into y to get $y = 2x/3 + 1$. This is a line with slope $2/3$ and y-intercept 1 .

65. For parametric equations, the slope of the graph is $dy/dx = (dy/dt)/(dx/dt)$. In this case, it is $3t^2/4t^3$. When $t = 1$, $dy/dx = 3/4$, $x = 1$, and $y = 2$. Thus, the tangent line is $(y - 2)/(x - 1) = 3/4$, i.e., $y = 3x/4 + 5/4$.

69. The length L is given by $L = \int_\alpha^\beta \sqrt{[r(\theta)]^2 + [r'(\theta)]^2}\,d\theta$. In this case, $r^2 + (r')^2 = \theta^4 + 4\theta^2 = \theta^2(\theta^2 + 4)$, and $\sqrt{r^2 + (r')^2}$ becomes $\theta\sqrt{\theta^2 + 4}$. Let $u = \theta^2 + 4$, so $du = 2\theta\,d\theta$, and $L = (1/2) \times \int_4^{\pi^2/4+4}\sqrt{u}\,du = (1/3)u^{3/2}\Big|_4^{\pi^2/4+4} = (1/3)[(\pi^2/4 + 4)^{3/2} - 8]$.

The area A is given by $A = (1/2)\int_\alpha^\beta [r(\theta)]^2\,d\theta$. In this case, $A = (1/2)\int_0^{\pi/2}\theta^4\,d\theta = (\theta^5/10)\big|_0^{\pi/2} = \pi^5/320$.

73. The length L is given by $L = \int_\alpha^\beta \sqrt{[r(\theta)]^2 + [r'(\theta)]^2}\,d\theta$. Here, $r' = 3\cos^3(\theta/4)\sin(\theta/4)$, so $r^2 + (r')^2 = 9\cos^8(\theta/4) + 9\cos^6(\theta/4)\sin^2(\theta/4) = 9\cos^6(\theta/4)[\cos^2(\theta/4) + \sin^2(\theta/4)] = 9\cos^6(\theta/4) = 9\cos^2(\theta/4)(1 -$

73. (continued)

$\sin^2(\theta/4))^2$. Then $L = 3\int_0^{\pi}[\cos(\theta/4) - \cos(\theta/4)\sin^2(\theta/4)]\,d\theta$. Let $u = \sin(\theta/4)$, so $du = (1/4)\cos(\theta/4)d\theta$, and $L = 12\int_0^{\sqrt{2}/2}(1 - u^2)du = 12(u - u^3/3)\Big|_0^{\sqrt{2}/2} = 12(\sqrt{2}/2 - \sqrt{2}/12) = 5\sqrt{2}$.

The area A is given by $A = (1/2)\int_{\alpha}^{\beta}[r(\theta)]^2 d\theta$. Thus, $A = (1/2) \times \int_0^{\pi} 9\cos^8(\theta/4)d\theta = (9/2)\int_0^{\pi}(1 + \cos(\theta/2))^4 d\theta/16 = (9/32)\int_0^{\pi}[1 + 4\cos(\theta/2) + 6\cos^2(\theta/2) + 4\cos^3(\theta/2) + \cos^4(\theta/2)]d\theta = (9/32)[\theta + 8\sin(\theta/2)]\Big|_0^{\pi} + (9/32)\int_0^{\pi}[6(1 + \cos\theta)/2 + 4\cos(\theta/2)(1 - \sin^2(\theta/2)) + (1 + \cos\theta)^2/4]d\theta = 9\pi/32 + 9/4 + (9/32)[3\theta + 3\sin\theta + 8\sin(\theta/2) - 8\sin^3(\theta/2)]\Big|_0^{\pi} + (9/128)\int_0^{\pi}(1 + 2\cos\theta + \cos^2\theta)d\theta = 9\pi/32 + 9/4 + 27\pi/32 + 9/4 - 9/4 + (9/128)(\theta + 2\sin\theta)\Big|_0^{\pi} + (9/256)\int_0^{\pi}(1 + \cos 2\theta)d\theta = 9\pi/8 + 9/4 + 9\pi/128 + (9/256)(\theta + \sin 2\theta/2)\Big|_0^{\pi} = 9\pi(1/8 + 1/128 + 1/256) + 9/4 = 315\pi/256 + 9/4$.

77. This exercise uses the formulas $\sin x \cos y = (1/2)[\sin(x + y) + \sin(x - y)]$ and $\cos x \cos y = (1/2)[\cos(x + y) + \cos(x - y)]$. $a_m = (1/\pi)\int_0^{2\pi}\cos 3x \cos mx\,dx = (1/2\pi)\int_0^{2\pi}[(\cos(3 + m)x + \cos(3 - m)x]dx = -(1/2\pi)[(1/(3 + m))\sin(3 + m)x + (1/(3 - m))\sin(3 - m)x]\Big|_0^{2\pi} = 0$ unless $m = 3$. Then $a_3 = (1/2\pi)\int_0^{2\pi}(\cos 6x + 1)dx = (1/2\pi)((1/6)\sin 6x + x)\Big|_0^{2\pi} = 1$. Also, $b_m = (1/\pi)\int_0^{2\pi}\cos 3x \sin mx\,dx = (1/2\pi)\int_0^{2\pi}[\sin(m + 3)x + \sin(m - 3)x]dx = -(1/2\pi)[(1/(m + 3))\cos(m + 3)x + (1/(m - 3))\cos(m - 3)x]\Big|_0^{2\pi} = 0$.

81. We use the product formulas listed on p. 460 and the half-angle formula $\sin^2 x = (1 - \cos 2x)/2$. $a_m = (1/\pi)\int_0^{2\pi}\sin^2 x \cos mx\,dx = (1/2\pi) \times \int_0^{2\pi}[\cos mx - \cos 2x \cos mx]dx = (1/2\pi)\int_0^{2\pi}[\cos mx - (1/2)(\cos(2 + m)x + \cos(2 - m)x)]dx = (1/4\pi)[(2/m)\sin mx - (1/(2 + m))\sin(2 + m)x - (1/(2 - m))\sin(2 - m)x]\Big|_0^{2\pi} = 0$, unless $m = 0$ or $m = 2$. Then $a_2 = -(1/4\pi) \times \int_0^{2\pi}[\cos 4x + 1]dx + 0 = -(1/4\pi)((1/4)\sin 4x + x)\Big|_0^{2\pi} = -1/2$; and $a_0 = (1/2\pi)\int_0^{2\pi}1\,dx + 0 = 1$. Also, $b_m = (1/\pi)\int_0^{2\pi}\sin^2 x \sin(mx)dx = (1/2\pi) \times$

81. (continued)

$\int_0^{2\pi}[\sin(mx) - \cos 2x \sin(mx)]\,dx = (1/2\pi)\int_0^{2\pi}[\sin(mx) - (1/2)(\sin(2 + m)x + \sin(m - 2)x]\,dx = (1/4\pi)[(2/m)\cos(mx) + (1/(2 + m))\cos(2 + m)x + (1/(m - 2))\cos(m - 2)x]\,\Big|_0^{2\pi} = 0$.

85. Substituting the given formulas yields $T = (4/\pi)\int_0^{\phi_m}(1 - 2\sin^2\phi - 1 + 2\sin^2\phi_m)^{-1/2}d\phi = (4/\pi)\int_0^{\phi_m}(-2\sin^2\phi_m\sin^2\beta + 2\sin^2\phi_m)^{-1/2}d\phi = (2\sqrt{2}/\pi)\times\int_0^{\phi_m}(1/\sin\phi_m)(1 - \sin^2\beta)^{-1/2}d\phi = (2\sqrt{2}/\pi)\int_0^{\phi_m}[1/(\sin\phi_m\cos\beta)]\,d\phi$. Differentiate $\sin\phi = \sin\phi_m\sin\beta$ to get $\cos\phi\,d\phi = \sin\phi_m\cos\beta\,d\beta$. Since $\cos\phi = \sqrt{1 - \sin^2\phi} = \sqrt{1 - k^2\sin^2\beta}$, where $k^2 = \sin^2\phi_m$, we get $d\phi = \sin\phi_m\cos\beta\,d\beta/\sqrt{1 - k^2\sin^2\beta}$. When $\phi = \phi_m$, $\sin\beta = 1$ so $\beta = \pi/2$; and when $\phi = 0$, $\beta = 0$. Substitute all this and cancel $\sin\phi_m\cos\beta$ to get $T = (4/\pi\sqrt{2})\int_0^{\pi/2}[1/\sqrt{1 - k^2\sin^2\beta}]\,d\beta$.

89. (a) The area beneath the graph is $A = \int_a^b(1 + x^n)dx = b - a + (b^{n+1} - a^{n+1})/(n + 1)$, unless $n = -1$. If $n = -1$, $A = \int_a^b(1 + x^{-1})dx = b - a + \ln(b/a)$.

(b) The length of this graph is $L = \int_a^b\sqrt{1 + n^2x^{2n-2}}\,dx$. If $n = 0$, then the length is $L = \int_a^b dx = b - a$. If $n = 1$, then $L = \int_a^b\sqrt{1 + 1}\,dx = \int_a^b\sqrt{2}\,dx = \sqrt{2}(b - a)$. If $n = 2$, then $L = \int_a^b\sqrt{1 + 4x^2}\,dx = [x\sqrt{1 + 4x^2} + (1/2)\ln|2x + \sqrt{1 + 4x^2}|]\,\Big|_a^b$. The $n = 2$ case is Example 3 of Section 10.3 .

For the case $n = 3/2$, we have $L = \int_a^b\sqrt{1 + (9/4)x}\,dx$. Substituting $u = 1 + (9/4)x$, we get $L = (8/27)(1 + 9x/4)^{3/2}\Big|_a^b = (1/27)(4 + 9x)^{3/2}\Big|_a^b$. For the more general case where $n = (2k + 3)/(2k + 2)$ and k is a nonnegative integer, we have $L = \int_a^b\sqrt{1 + [(2k + 3)/(2k + 2)]^2x^{1/(k+1)}}\,dx$. Now, let $u = n^2x^{2n-2}$ so $x = n^{1/(1-n)}(u - 1)^{1/(2n-2)}$ and $dx = n^{1/(1-n)}du/[(2n - 2)\times(u - 1)^{(3-2n)/(2n-2)}]$. Then $L = \int_{1+n^2a^{2n-2}}^{1+n^2b^{2n-2}}[\sqrt{u}n^{1/(1-n)}du/$

89. (b) (continued)

$(2n-2)(u-1)^k]$. Recall that the binomial expansion for $(u-1)^k$ is $\Sigma_{i=0}^{k}\binom{k}{i}u^i$ and this gives us $L = (n^{1/(1-n)}/(2n-2))\times \int_{1+n^2a^{2n-2}}^{1+n^2b^{2n-2}} \Sigma_{i=0}^{k}\binom{k}{i}(-1)^{k-i}u^{i+1/2}du = (n^{1/(1-n)}/(2n-2))\times$ $\Sigma_{i=0}^{(3-n)/(2n-2)}\binom{(3-n)/(2n-2)}{i}(-1)^{(3-n)/(2n-2)-i}[(1+n^2b^{2n-2})^{i+3/2} - (1+n^2a^{2n-2})^{i+3/2}]/(i+3/2)$. (Note: recall that $\binom{k}{i} = k!/i!(k-i)!$) .

(c) Around the x-axis, $V_x = \pi\int_a^b(1+2x^n+x^{2n})dx = \pi[b-a+2(b^{n+1}-a^{n+1})/(n+1)+(b^{2n+1}-a^{2n+1})/(2n+1)]$, unless $n=-1$ or $n=-1/2$. If $n=-1$, $V_x = \pi\int_a^b(1+2x^{-1}+x^{-2})dx = \pi[b-a+2\ln(b/a)-(a^{-1}-b^{-1})]$. If $n=-1/2$, $V_x = \pi\int_a^b(1+2x^{-1/2}+x^{-1})dx = \pi[b-a+4\sqrt{b}-4\sqrt{a}+\ln(b/a)]$. Around the y-axis, $V_y = 2\pi\int_a^b(x+x^{n+1})dx = \pi[b^2-a^2+2(b^{n+2}-a^{n+2})/(n+2)]$ unless $n = -2$. If $n=-2$, $V_y = 2\pi\int_a^b(x+x^{-1})dx = \pi[b^2-a^2+2\ln(b/a)]$.

89. (d)

Around the x-axis, $A_x = 2\pi\int_a^b(1+x^n)\sqrt{1+n^2x^{2n-2}}\,dx = 2\pi\int_a^b\sqrt{1+n^2x^{2n-2}}\,dx + 2\pi\int_a^b x^n\sqrt{1+n^2x^{2n-2}}\,dx$. Note that $2\pi\int_a^b\sqrt{1+n^2x^{2n-2}}\,dx$ is 2π times the arc length which was analyzed in part (b). Thus, we only need to look at $a_x = 2\pi\int_a^b x^n\sqrt{1+n^2x^{2n-2}}\,dx$.

If $n=0$, then $a_x = 2\pi\int_a^b dx = 2\pi(b-a)$. If $n=1$, then $a_x = 2\pi\int_a^b\sqrt{2}x\,dx = \sqrt{2}\pi(b^2-a^2)$.

If $n=2$, then $a_x = 2\pi\int_a^b(x^2\sqrt{1+4x^2})dx$. The integral $\int_a^b(x^2\sqrt{1+4x^2})dx$ is equal to $\int_a^b[(x^2+4x^2)/\sqrt{1+4x^2}]dx = \int_a^b(x^2/\sqrt{1+4x^2})dx$ $\int_a^b(4x^4/\sqrt{1+4x^2})dx$. The latter integral is integrated by parts with $u=x^3$, $dv = 4x/\sqrt{1+4x^2}\,dx$, $du = 3x^2\,dx$, and $v=\sqrt{1+4x^2}$. Therefore,

89. (d) (continued)

$\int_a^b (x^2\sqrt{1+4x^2})dx = \int_a^b (x^2/\sqrt{1+4x^2})dx + [x^3\sqrt{1+4x^2}|_a^b - 3\int_a^b (x^2\sqrt{1+4x^2})dx]$. Rearrangement yields $4\int_a^b (x^2\sqrt{1+4x^2})dx = b^3\sqrt{1+4b^2} - a^3\sqrt{1+4a^2} + \int_a^b (x^2/\sqrt{1+4x^2})dx$. The latter integral can be done with a trigonometric substitution. Let $x = (1/2)\tan\theta$ so $dx = (1/2)\sec^2\theta d\theta$. This yields $(1/8)\int_{\tan^{-1}2a}^{\tan^{-1}2b} \tan^2\theta \sec\theta\, d\theta = (1/8)\int_{\tan^{-1}2a}^{\tan^{-1}2b} (\sec^3\theta - \sec\theta)d\theta$. The integration of $\sec\theta$ and $\sec^3\theta$ are shown in detail in Example 6(b), Section 10.1 and Example 3, Section 10.3, respectively. Therefore, $\int_a^b (x^2/\sqrt{1+4x^2})dx = (1/16)[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| - 2\ln|\sec\theta + \tan\theta|]\,|_{\tan^{-1}2a}^{\tan^{-1}2b}$. Now if $\tan\theta = 2b$, then $\sec\theta = \sqrt{1+2b^2}$ and similarly, if $\tan\theta = 2a$, then $\sec\theta = \sqrt{1+2a^2}$. This substitution gives us $(1/16)(2b\sqrt{1+4b^2} - \ln|\sqrt{1+4b^2} + 2b| - 2a\sqrt{1+4a^2} + \ln|\sqrt{1+4a^2} + 2a|)$. Finally, we get $a_x = (\pi/2)[(1/16)\ln|(\sqrt{1+4a^2} + 2a)/(\sqrt{1+4b^2} + 2b)| + b\sqrt{1+4b^2}(b^2 + 1/8) - a\sqrt{1+4a^2}(a^2 + 1/8)]$.

If $n = (2k+3)/(2k+2)$ where k is any nonnegative integer, let $x = ((u-1)/n^2)^{1/(2n-2)}$ so $dx = 1/[((2n-2)n^{1/(n-1)}) \times (u-1)^{(3-2n)/(2n-2)}du$. Substitute this into $a_x = 2\pi\int_a^b x^n\sqrt{1+n^2x^{2n-2}}dx = [\pi/(n-1)n^{(n+1)/(n-1)}]\int_{x=a}^{x=b} (u-1)^{(3-n)/(2n-2)}\, u\, du$. Substitute for n in $(3-n)/(2n-2) = ((3-1-2)/(2k+1))/(2+4/(2k+1)-2) = (4k+2-2)/4 = k$. Hence $a_x = [\pi n^{(n+1)/(1-n)}/(n-1)]\int_{x=a}^{x=b}(u-1)^k\sqrt{u}\, du$. From the binomial expansion formula, $(u-1)^k = \Sigma_{i=0}^k \binom{k}{i}(-1)^{k-i}u^i$. Let $I = \int(u-1)^k\sqrt{u}\, du$. Then $I = \int\Sigma_{i=0}^k \binom{k}{i}(-1)^{k-i}u^{i+1/2}\, du = \Sigma_{i=0}^k \binom{k}{i}(-1)^{k-i}u^{i+3/2}/(i+3/2) + C$. Substitute for u and k to get $a_x = \pi n^{(n+1)/(1-n)}/(n-1)\left|\Sigma_{i=0}^{(3-n)/(2n-2)}\binom{(3-n)/(2n-2)}{i} \times (-1)^{(3-n)/(2n-2)-i}(1+n^2x^{2n-2})^{i+3/2}/(i+3/2)\right|\Big|_a^b = [\pi n^{(n+1)/(1-n)}/(n-1)]\Sigma_{i=0}^{(3-n)/(2n-2)}\binom{(3-2n)/(2n-2)}{i}(-1)^{(3-n)/(2n-2)-i} \times ((1+n^2b^{2n-2})^{i+3/2} - (1+n^2a^{2n-2})^{i+3/2})/(i+3/2)$.

89. (d) (continued)

Around the y-axis, $A_y = 2\pi\int_a^b x\sqrt{1 + n^2x^{2n-2}}\,dx$. If $n = 0$, then $A_y = 2\pi\int_a^b x\,dx = \pi(b^2 - a^2)$. If $n = 1$, then $A_y = 2\pi\int_a^b \sqrt{2}x\,dx = \pi\sqrt{2}(b^2 - a^2)$. If $n = 2$, then substitution of $u = 1 + 4x^2$ yields $A_y = 2\pi\int_a^b x\sqrt{1 + 4x^2}\,dx = (\pi/6)(1 + 4x^2)^{3/2}\big|_a^b = (\pi/6)[(1 + 4b^2)^{3/2} - (1 + 4a^2)^{3/2}]$.

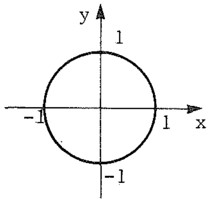
If $n = 3$, then $A_y = 2\pi\int_a^b x\sqrt{1 + 9x^4}\,dx$. Let $u = 3x^2$ so $du = 6x\,dx$. Then $A_y = (\pi/3)\int_{3a^2}^{3b^2} \sqrt{1 + u^2}\,du$ Integrate as in Example 3, Section 10.3 to get $A_y = (\pi/3)[(u/2)\sqrt{u^2 + 1} + (1/2)\ln|u + \sqrt{u^2 + 1}|]\big|_{3a^2}^{3b^2} = (\pi/6)(3b^2\sqrt{9b^4 + 1} + \ln[(3b^2 + \sqrt{9b^4 + 1})/(3a^2 + \sqrt{9a^4 + 1})] - 3a^2\sqrt{9a^4 + 1}$.

If $n = (k + 3)/(k + 2)$ where k is any nonnegative integer, then $(2 - n)/(n - 1) = (2 - 1 - 1/(k + 1))/(1/(k + 1)) = (k + 1 - 1)/1 = k$. Let $u = 1 + n^2x^{2n-2}$, so $x = (u - 1)^{1/(2n-2)}n^{1/(1-n)}$ and $dx = (n^{1/(1-n)}(u - 1)^{(3-2n)/(2n-2)}/(2n - 2))\,du$. Then $A_y = [\pi n^{2/(1-n)}/(n - 1)]\int_{1+n^2a^{2n-2}}^{1+n^2b^{2n-2}} (u - 1)^{(2-n)/(n-1)}\sqrt{u}\,du$. Since $(2 - n)/(n - 1) = k$, expand $(u - 1)^k$ with the binomial formula: $(u - 1)^k$ $\Sigma_{i=0}^{k}\binom{k}{i}(-1)^{k-i}u^i$. Substitute this into the integral to get $A_y = \pi[n^{2/(1-n)}/(n - 1)]\Sigma_{i=0}^{(2-n)/(n-1)}\binom{(2-n)/(n-1)}{i}[(-1)^{(2-n)/(n-1)-1}/(i + 3/2)] \times [(1 + n^2b^{2n-2})^{i+3/2} - (1 + n^2a^{2n-2})^{i+3/2}]$.

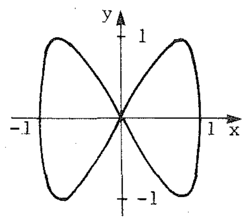
93. (a) Plot the curves point-by-point, but notice that if $m = 1$ and n is any integer, then for $t = 0$, the point P is $(1,0)$; for $t = \pi$, P is $(-1,0)$. Also, since $\cos(\pi - \alpha) = \cos(\pi + \alpha)$ and $\sin(\pi - \alpha) = -\sin(\pi + \alpha)$ for all α, the curves are symmetric about the x-axis. Since $\cos(\pi - \alpha) = -\cos\alpha$ and $\sin(\pi - \alpha) = \sin\alpha$, the curves are also symmetric about the y-axis and there-

93. (a) (continued)

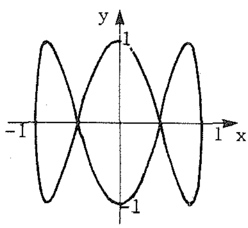
fore, the origin. Finally, since $y(\pi/n) = 0$, each curve crosses the x-axis at $n + 1$ points. (Actually, each curve crosses $2n + 2$ times, but as t goes from π to 2π the curve will recross the x-axis at the same points it crossed when t went from 0 to π.)



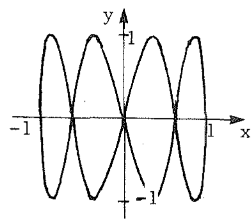
m=1, n=1



m=1, n=2



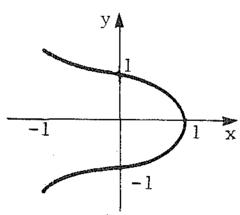
m=1, n=3



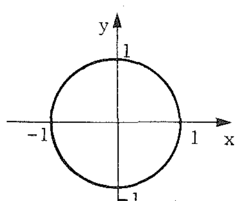
m=1, n=4

(b) Consequently, each curve will consist of n loops, for n odd or even.

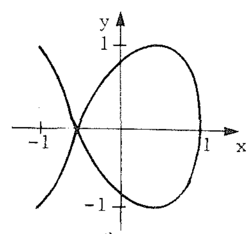
(c)



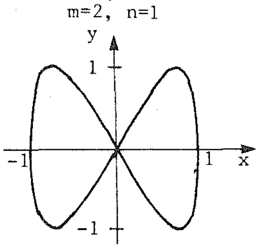
m=2, n=1



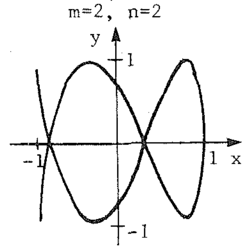
m=2, n=2



m=2, n=3

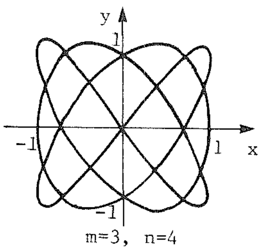


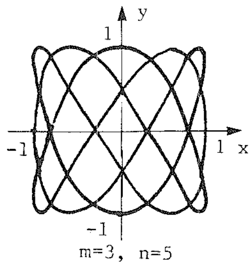
m=2, n=4



m=2, n=5

93. (d)





TEST FOR CHAPTER 10

1. True or false.

   (a) Every polynomial is a product of linear and/or quadratic factors.

   (b) The area between $r = f(\theta)$ and $r = g(\theta)$, where $f(\theta) \geqslant g(\theta)$ on the interval $[\alpha,\beta]$, is $(1/2)\int_\alpha^\beta [f(\theta) - g(\theta)]^2$.

   (c) Using the method of partial fractions, $1/(x - 1)^2x^2 = A/(x - 1) + B/(x - 1)^2 + (Cx + D)/x^2$ for some constants A, B, C, and D.

   (d) If $f(x) \geqslant 0$ on $[a,b)$, the surface area obtained by revolution around the x-axis is $2\pi\int_a^b \sqrt{[f(x)]^2 + [f'(x)f(x)]^2}\,dx$.

   (e) In polar coordinates, the arc length of $r = f(\theta)$ over the interval $[\alpha,\beta]$ is $\int_\alpha^\beta \sqrt{1 + [f'(\theta)]^2}\,d\theta$.

2. The line $y = -x + 1$ on $[0,3]$ is revolved to form a surface of revolution. Find the surface area if the line is revolved around:

   (a) the y-axis

   (b) the x-axis

3. Find the arc length of the curves described by the following equations on the given intervals:

   (a) $y = (2/3)x^{3/2}$ on $[0,8]$

   (b) $r = \sin^2\theta$ in polar coordinates for $0 \leqslant \theta \leqslant 2\pi$

   (c) $y = t^2 + 3$ and $x = 2t + 1$ for $0 \leqslant t \leqslant 1$

4. Evaluate the following integrals:

   (a) $\int_0^1 [(x^3 + 7x^2 - 8)/(x^2 - 4)]\,dx$

   (b) $\int [dt/t(t^2 - t + 1)]$

5. Show that $\int_0^{n\pi/2} \sin^2\theta\, d\theta = \int_0^{n\pi/2} \cos^2\theta\, d\theta = n\pi/4$ for all integers $n \geqslant 0$.

6. Suppose $y_1 = f(x)$ and $y_2 = f(x) + 3$. If $f(x) \geqslant 0$ on $[a,b]$, where $b > a > 0$, then which of the following are equal, if any?

   (a) The arc lengths of y_1 and y_2 on $[a,b]$.

   (b) The surface areas of revolution obtained by revolving y_1 and y_2 on $[a,b]$ around the x-axis.

   (c) The surface areas of revolution obtained by revolving y_1 and y_2 on $[a,b]$ around the y-axis.

7. Compute the average value of $f(t)$ for the given interval:

   (a) $f(t) = 1/(t - 1)^2 t$ on the interval $2 \leqslant t \leqslant 3$.

   (b) $f(t) = \sin^2(t/2)\cos^3(t/2)$ on the interval $0 \leqslant t \leqslant \pi/2$.

8. Find the area of the region inside $r = \sin\theta\cos\theta$ and outside $r = \cos 2\theta$.

9. Find the area between the following curves and the x-axis in the given intervals:

   (a) $y = \sin^4 x$ on $[0, \pi/8]$

   (b) $y = x \sin^2 x \cos x$ on $[\pi/4, \pi/2]$

10. Dragster Debbie, the 80-year-old grandmother, needed some excitement in her life. She decided to take her Ferrari for a little spin down at the speedway. As she floors the gas petal, her position can be described parametrically by $x = 2\cos t$ and $y = 4\sin t\cos t$.

    (a) What is her speed at time t_0 ?

    (b) At $t = 8\pi$, she realized she had to hurry to her karate lessons, so she sped off along the tangent line. Describe the tangent line with a set of parametric equations.

ANSWERS TO CHAPTER TEST

1. (a) True

   (b) False; it is $(1/2)\int_{\alpha}^{\beta}\{[f(\theta)]^2 - [g(\theta)]^2\}d\theta$.

   (c) False; $(Cx + D)/x^2$ should be $C/x + D/x^2$.

   (d) True

   (e) False; the given formula is for cartesian coordinates

2. (a) $9\pi\sqrt{2}$

   (b) $5\pi\sqrt{2}$

3. (a) $52/3$

   (b) $4 + (2\sqrt{3}/3)\ln(\sqrt{3} + 2)$

   (c) $\sqrt{2} + \ln(1 + \sqrt{2})$

4. (a) $15/2 - 3\ln 3 - 4\ln 2$

   (b) $\ln|t/\sqrt{t^2 - t + 1}| + (1/\sqrt{3})\tan^{-1}[(2t - 1)/3] + C$

5. $\int_0^{n\pi/2}\sin^2\theta = \int(1/2 - \cos 2\theta/2)d\theta = (\theta/2 - \sin 2\theta/4)\Big|_0^{n\pi/2} = n\pi/4$;
   $\int_0^{n\pi/2}\cos^2\theta = \int(1/2 + \cos 2\theta/2)d\theta = (\theta/2 + \sin 2\theta/4)\Big|_0^{n\pi/2} = n\pi/4$.

6. (a) and (c)

7. (a) $\ln(3/4) + 1/2$

   (b) $7\sqrt{2}/30\pi$

8. $\pi - 4/5$

9. (a) $3\pi/64 - \sqrt{2}/8 + 1/32$

   (b) $(3\pi\sqrt{2} + 20\sqrt{2} - 24\pi)/144$

10. (a) $2[\sin^2 t_0 + 4\sin^4 t_0 + 4\cos^4 t_0 - 2\cos^2 t_0\sin^2 t_0]^{1/2}$

    (b) $x = 2$ and $y = 4(t - 8\pi)$